Constructing compacta from relations between finite graphs

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Outline

- We consider a construction assigning to every poset $\mathbb{P}$ a compact space $\mathrm{S}\mathbb{P}$, the spectrum.
- The intention is that elements $p \in \mathbb{P}$ correspond to basic open sets of $\mathrm{S}\mathbb{P}$, and the order relation corresponds to containment.
- We may also start with a compact space X and its open basis $\mathbb{P}$, and ask when X can be reconstructed as $\mathrm{S}\mathbb{P}$.

Theorem

Every metrizable compact space X can be reconstructed from the poset structure of any countable basis $\{B_n : n \in \omega\}$ such that $\operatorname{diam}(B_n) \to 0$.

Outline

- An abstract poset $\mathbb{P}$ is often obtained from a sequence of finite graphs $(G_n, \sqcap_n)$ and bonding relations $\sqsubset_n \colon G_{n+1} \to G_n$ by taking the disjoint union $\bigsqcup_n G_n$ and letting $\leq$ be the transitive closure of $\bigsqcup_n \sqsubset_n$.
- Such posets are called graded ω-posets.
- The intention is that $p \sqcap q$ in G_n iff the corresponding basic open sets in $\mathrm{S}\mathbb{P}$ overlap.
- Then we can reason about $\mathrm{S}\mathbb{P}$ using directly the properties of the finite graphs $(G_n, \sqcap_n)$ and of the bonding relations $\sqsubset_n$.

Comparison to inverse sequences

- A more common approach to build compact spaces from inverse sequences of graphs and homomorphisms is to consider quotients of inverse limits.
- The inverse limit approach is used in the context of projective Fraïssé theory introduced by Irwin and Solecki (which is one of our motivations), and also in the context of constructing topologically complete spaces from so called cell structures (Dębski and Tymchatyn).
- Given a an inverse sequence $G_0 \xleftarrow{f_0} G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} \cdots$ of finite graphs $\langle G_n, \sqcap \rangle$ and $\sqcap$-preserving maps, the inverse limit is the compact space $G_\infty := \{x \in \prod_n G_n : \forall n \; x_n = f_n(x_{n+1})\}$ together with the closed relation $x \sqcap y \;:\iff\; \forall n \, x_n \sqcap y_n$.
- If the relation $\sqcap$ turns out to be transitive, the quotient $G_\infty / \sqcap$ is a compact metrizable space.

Comparison to inverse sequences

- Given a an inverse sequence $G_0 \overset{f_0}{\leftarrow} G_1 \overset{f_1}{\leftarrow} G_2 \overset{f_2}{\leftarrow} \cdots$ of finite graphs $\langle G_n, \sqcap \rangle$ and $\sqcap$-preserving maps, the inverse limit is the compact space $G_\infty := \{x \in \prod_n G_n : \forall n \; x_n = f_n(x_{n+1})\}$ together with the closed relation $x \sqcap y \; :\Longleftrightarrow \; \forall n \, x_n \sqcap y_n$.
- If the relation $\sqcap$ turns out to be transitive, the quotient $G_\infty/\sqcap$ is a compact metrizable space.
- It is typically not the case that $G_\infty/\sqcap \cong \mathrm{S}\mathbb{P}$ where $\mathbb{P} := \bigsqcup_n G_n$ is the induced poset.
- However, we may put G'_n to be the set of all maximal cliques of size ≤ 2 in G_n (i.e. edges and isolated points), $\sqsubset' \colon G'_{n+1} \to G'_n$ to be the refinement relation, and $\mathbb{P}' := \bigsqcup_n G'_n$.

Theorem

If every element of G_∞ is $\sqcap$-related to at most one other element, then $G_\infty/\sqcap \cong \mathrm{S}\mathbb{P}'$.

Application: Fraïssé limits

- Our goal is to represent certain well-known spaces as Fraïssé limits of the following form.
- We take a category $\mathcal{C}$ of desired graphs and desired co-bijective relational morphisms.
- We prove that the category $\mathcal{C}$ is directed and has the amalgamation property, so that there exists an essentially unique Fraïssé sequence $((G_n), \sqsubset_n)$.
- We consider the induced atomless graded ω-poset $\mathbb{P} := \bigsqcup_n G_n$ and its spectrum $\mathsf{S}\mathbb{P}$.
- We prove that "nice" morphisms form a wide ideal in $\mathcal{C}$, so there is a Fraïssé sequence consisting of "nice" morphisms, and so $\mathbb{P}$ is well behaved.
- Namely, $\mathsf{S}\mathbb{P}$ is Hausdoff, $\mathbb{P}$ faithfully represents its basis, and overlaps of basic open sets correspond to the edge relations $\sqcap$.
- We examine further properties of $\mathsf{S}\mathbb{P}$ coming from $\mathcal{C}$, and using a suitable topological characterization we identify the space $\mathsf{S}\mathbb{P}$.

Application: Fraïssé limits – examples

graphs	co-bijective morphisms	$S\mathbb{P}$
discrete	all ($\Leftrightarrow$ surjective functions)	Cantor space
paths	monotone	arc
paths	all	pseudo-arc
fans	root-monotone end-preserving	Cantor fan
fans	root-monotone	Lelek fan

[We are still working on proofs of some of these.]

Our construction

1. (abstract basic) open sets
   - We start with a poset $\mathbb{P}$.
2. (abstract) open covers
   - For $B, C \subseteq \mathbb{P}$ we write $B \leq C$ for $\forall b \in B\ \exists c \in C\ b \leq c$.
   - A band is a finite set $B \subseteq \mathbb{P}$ such that for every $p \in \mathbb{P}$ there is $b \in B$ with $p \leq b$ or $b \leq p$.
   - A cap is a set $C \subseteq \mathbb{P}$ such that $B \leq C$ for some band B.
3. points
   - A selector is a set $S \subseteq \mathbb{P}$ such that $S \cap C \neq \emptyset$ for every cap C.
   - The spectrum $\mathrm{S}\mathbb{P}$ consists of $\subseteq$-minimal selectors.
   - Note that every selector contains a minimal selector.
4. topology
   - For every $p \in \mathbb{P}$ we put $p^{\in} := \{S \in \mathrm{S}\mathbb{P} : p \in S\}$.
   - We endow $\mathrm{S}\mathbb{P}$ with the topology generated by the sets $p^{\in}$.

Properties of the spectrum

For any poset $\mathbb{P}$:

- $\mathrm{S}\mathbb{P}$ is a compact T_1 space.
- Every minimal selector S is upwards closed.
- $\{p^{\in} : p \in \mathbb{P}\}$ is an open subbasis for $\mathrm{S}\mathbb{P}$.
- Every set $C \subseteq \mathbb{P}$ is a cap iff $\{c^{\in} : c \in C\}$ covers $\mathrm{S}\mathbb{P}$.

For $\mathbb{P}$ an ω-poset:

- $\mathrm{S}\mathbb{P}$ is second countable.
- Every two caps C, C' have a common refinement $D \leq C, C'$.
- Every minimal selector S is a filter.
- $\{p^{\in} : p \in \mathbb{P}\}$ is an open basis for $\mathrm{S}\mathbb{P}$.

For $\mathbb{P}$ a regular ω-poset (defined later):

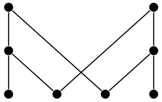
- $\mathrm{S}\mathbb{P}$ is Hausdorff, and so metrizable.

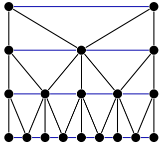
Examples

- For $\mathbb{P} = 2^{<\omega}$ (the full binary tree), $S\mathbb{P}$ is the Cantor space.
- For $\mathbb{P}$ consisting of a chain $p_0 > p_1 > \cdots$ together with an antichain $p_n > q_{n+1}$, $n \in \omega$, $S\mathbb{P}$ is the convergent sequence $\omega + 1$.
- For $\mathbb{P}$ consisting of two chains $p_0 > p_1 > \cdots$, $p'_0 > p'_1 > \cdots$ together with an antichain $p_n, p'_n > q_{n+1}$, $n + 1$, $S\mathbb{P}$ is the non-Hausdorff convergent sequence with two limits.
- For $\mathbb{P} = \mathbb{B} \setminus \{0\}$ where $\mathbb{B}$ is an atomless Boolean algebra, $S\mathbb{P}$ is just a singleton – bands are exactly finite subsets containing the top element.
- For $\mathbb{P} = \mathbb{B} \setminus \{0\}$ where $\mathbb{B}$ is the finite-cofinite algebra on $\mathcal{P}(\kappa)$, $S\mathbb{P}$ is the one-point compactification of the discrete space κ.

ω-Posets

- For $p \in \mathbb{P}$ we define the rank $\mathrm{r}(p) := \sup_{q>p}(\mathrm{r}(q)+1)$.
- $\mathbb{P}$ is an ω-poset if
  - $\mathrm{r}(p)$ is well-defined and finite for every $p \in \mathbb{P}$,
  - every set $\{p \in \mathbb{P}, \mathrm{r}(p) \leq n\}$ is finite.
- An ω-poset is graded if for every $p \leq q$ and $n \in [\mathrm{r}(q), \mathrm{r}(p)]$ there is $r \in [p, q]$ with $\mathrm{r}(r) = n$.
- Graded ω-posets come from sequences of sets $(G_n)_n$ with total relations $\sqsubset_n \colon G_{n+1} \to G_n$ by putting $\mathbb{P} := \bigsqcup_n G_n$ and letting $\leq$ be the transitive closure of $\bigsqcup_n \sqsubset_n$.
  - Then we have $\mathrm{r}(p) = n$ iff $p \in G_n$.
  - We may turn G_ns into graphs by putting $p \sqcap q$ if $\exists r \leq p, q$.
  - Then the relations $\sqsubset_n$ are $\sqcap$-preserving and reflecting.





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Reconstruction of spaces

- A cap-basis of a (necessarily compact) T_1 space X is a basis $\mathbb{P}$ such that every $C \subseteq \mathbb{P}$ that covers X is a cap (the other implication always holds).

Theorem

- If $\mathbb{P}$ is a cap-basis of a T_1 space X, then $x \mapsto \{p \in \mathbb{P} : x \in p\}$ is a homeomorphism $X \to \mathrm{S}\mathbb{P}$.

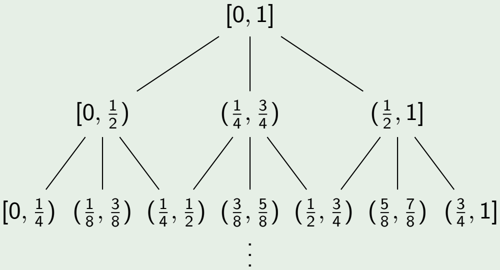
- Every second countable compact T_1 space X has a cap-basis $\mathbb{P}$ that is a graded ω-poset.
- Every second countable compact T_1 arises as a spectrum.

Reconstruction of spaces

Theorem

Let $\mathbb{P} = \{B_n : n \in \omega\}$ be a basis of non-empty open subsets of a compact metrizable space X. Then $\mathbb{P}$ is a cap-basis and an ω-poset if and only if $\mathrm{diam}(B_n) \to 0$.

Example



Regularity

- $p \leq q$ vs. $p^{\in} \subseteq q^{\in}$:
  - $\Rightarrow$ is always true,
  - $\Leftarrow$ holds iff $\mathbb{P}$ is cap-determined, i.e. for every $p \not\leq q$ there is a (finite) cap C such that $(C \setminus \{p\}) \cup \{q\}$ is not a cap.
  - Every cap-basis is a cap-determined poset.
- $\exists r \leq p, q$ vs. $\exists S \in p^{\in} \cap q^{\in}$:
  - $\Leftarrow$ holds if $\mathbb{P}$ is an ω-poset,
  - $\Rightarrow$ holds if $\mathbb{P}$ is prime, i.e. for every p there is a (finite) cap C such that $C \setminus \{p\}$ is not a cap, equivalently $p^{\in} \neq \emptyset$.
  - Every cap-determined $\mathbb{P}$ is prime.
- $p \lhd q$ vs. $\mathrm{cl}(p^{\in}) \subseteq q^{\in}$:
  - We define $p \lhd q$ if $Cp \leq q$ for some cap C where Cp is the star $\{c \in C : \exists r \leq c, p\}$.
  - $\Rightarrow$ holds if $\mathbb{P}$ is an ω-poset,
  - $\Leftarrow$ holds if $\mathbb{P}$ is a prime ω-poset.
- $\mathbb{P}$ regular vs. $\mathsf{S}\mathbb{P}$ Hausdorff:
  - We say that $\mathbb{P}$ is regular if every cap C is $\lhd$-refined by a cap D.
  - $\Rightarrow$ holds if $\mathbb{P}$ is an ω-poset,
  - $\Leftarrow$ holds if $\mathbb{P}$ is a prime ω-poset.

Categories of graphs

- A graph is a finite set G with a symmetric reflexive relation $\sqcap$.
- A morphism $\sqsubset \colon G \to H$ is a $\sqcap$-preserving relation.
- A co-bijective morphism $\sqsubset$ is
  1. co-surjective ("total"): $\forall g \in G,\ \exists h \in H\colon\ g \sqsubset h$,
  2. surjective: $\forall h \in H,\ \exists g \in G\colon\ g \sqsubset h$,
  3. co-injective ("subfunctional"): $\forall h \in H,\ \exists g \in G\colon\ g^{\sqsubset} = \{h\}$.
- A "nice" morphism $\sqsubset$ is co-bijective and
  4. anti-injective: $\forall h \in H\colon\ |h^{\sqsupset}| \geq 2$,
  5. star-refinining: $\forall g \in G,\ \exists h \in H\colon\ g' \sqcap g \Rightarrow g' \sqsubset h$,
  6. witnessing: $\forall h \sqcap h' \in H,\ \exists g \in H\colon\ g \sqsubset h, h'$.
- An inverse sequence $((G_n), \sqsubset_n)$ in a graph category is turned into an ω-poset $\mathbb{P} := \bigsqcup_n G_n$ where $\leq$ is generated by $\bigsqcup_n \sqsubset_n$.
  - 1 assures that $\mathbb{P}$ is graded,
  - 2 assures that $\mathbb{P}$ is atomless,
  - 3 and 4 assure that $\mathbb{P}$ is cap-determined,
  - 5 assures that $\mathbb{P}$ is regular,
  - 6 assures that the graph structure is reconstructed from $\mathbb{P}$.

Further research: maps and dynamics

- We have obtained some projetive Fraïssé limits as spectra of graded ω-posets coming from Fraïssé sequences of finite graphs and bonding relations.
- How about generic automorphisms?
- A refiner is a relation $\sqsubset : \mathbb{P} \to \mathbb{Q}$ such that every cap in $\mathbb{Q}$ is refined by a cap in $\mathbb{P}$.

Theorem

If $\sqsubset : \mathbb{P} \to \mathbb{Q}$ and $\sqsubset' : \mathbb{Q} \to \mathbb{P}$ are refiners such that $\sqsubset' \circ \sqsubset \subseteq \leq_{\mathbb{P}}$ and $\sqsubset \circ \sqsubset' \subseteq \leq_{\mathbb{Q}}$, then $\mathsf{S}\mathbb{P} \cong \mathsf{S}\mathbb{Q}$.

Thank you.